EVERY TREE IS A LARGE SUBTREE OF A TREE THAT DECOMPOSES K_n OR $K_{n,n}$

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Abstract. Let T be a tree with m edges. A well-known conjecture of Ringel states that every tree T with m edges decomposes the complete graph K_{2m+1}. Graham and Häggkvist conjectured that T also decomposes the complete bipartite graph $K_{m,m}$. In this paper, we show that there exists an integer n with $n \leq \lceil 3(m-1)/2 \rceil$ and a tree T_1 with n edges such that T_1 decomposes K_{2n+1} and contains T. We also show that there exists an integer n' with $n' \geq 2m-1$ and a tree T_2 with n' edges such that T_2 decomposes $K_{n',n'}$. In the latter case, we can improve the bound if there exists a prime p such that $\lceil 3m/2 \rceil \leq p \leq 2m-1$.

Keywords Graph labelings, Graph decompositions, Combinatorial nullstellensatz

1. Introduction

An *decomposition* of a graph G is a partition $\mathcal{P}$ of its set of edges. When the graph induced by each part of $\mathcal{P}$ is isomorphic to a graph H, we say that H *decomposes* G and write $H|G$.

A famous conjecture of Ringel from 1963 states that every tree with m edges decomposes the complete graph K_{2m+1}. In spite of the hundreds of papers which have appeared in the literature (see the dynamic survey of Gallian [5]), Ringel's conjecture is still wide open. Graham and Häggkvist proposed the following generalization of Ringel's conjecture; see e.g. [6]:

Conjecture 1 (Graham and Häggkvist). *Every tree with m edges decomposes every $2m$-regular graph and every bipartite m-regular graph.*

Conjecture 1 in particular asserts that every tree with m edges decomposes the complete bipartite graph $K_{m,m}$. In the sequel we will refer to this particularization of Conjecture 1.

Both conjectures are known to hold for caterpillars, for trees of diameter at most five and for various particular families of trees.

In one of the early papers on the subject, Kotzig [10] showed that the substitution of an edge by a sufficiently large path in an arbitrary tree results in a tree T which verifies Ringel's conjecture. Thus every tree is homeomorphic to a tree which verifies this conjecture. On the other hand Kézdy [8] showed that the addition of an unspecified number of leaves to a vertex of a tree results in a tree with n edges which decomposes K_{2n+1}. An analogous result for the decomposition of $K_{n,n}$ had been also proved in [11]. Therefore, every tree contains the base tree of some tree verifying both conjectures. However, neither result gives a quantitative estimate of the number of additional vertices that will suffice to make a tree decompose the appropriate complete graph.

In this paper we consider an approximation to both conjectures and prove that every tree is a large subtree of two trees for which the conjectures hold respectively. We prove:

Theorem 1. *Let T be a tree with m edges.*

(i) *For every odd $n \geq 2m-1$ there exists a tree T' with n edges that decomposes $K_{n,n}$ and contains T.*

(ii) *For every prime $p \geq \lceil 3m/2 \rceil$ there exists a tree T' with p edges that decomposes $K_{p,p}$ and contains T.*

Theorem 2. *Let T be a tree with m edges. For every $n \leq \lceil 3(m-1)/2 \rceil$, there exists a tree T' with at n edges that decomposes K_{2n+1} and contains T.*

2. The tools

The classical approach to the decomposition problem uses labeling techniques that aim to find *cyclic* decompositions. A tree T with m edges cyclically decomposes K_{2m+1} if there is an injection $\phi : V(T) \to [0, 2m]$ such that the translations $\phi(v) + k \pmod{2m+1}$ give $(2m+1)$ edge-disjoint copies of T. Similarly, T cyclically decomposes $K_{m,m}$ if there is a map $\phi : V(T) \to [0, m-1]$ that is injective on each partite set of T such that the translations $\phi(v) + k \pmod{m}$ produce m edge-disjoint copies of T.

A ρ-valuation of a graph H on m edges is an injection $\rho: V(H) \to \mathbb{Z}_{2m+1}$ so that the induced edge labels $\rho_E(uv) := \rho(u) - \rho(v)$, for $uv \in E(H)$, satisfy

$$\rho_E(e) \neq \pm\rho_E(f) \pmod{2m+1},$$

for all distinct pairs of edges $e, f \in E(H)$. Rosa [13] proved that a graph H with m-edges cyclically decomposes K_{2m+1} if an only if it admits a ρ-valuation.

Similarly, a modular bigraceful labeling of a bipartite graph H with m edges and partite sets A, B is a map $f : V(H) \to \mathbb{Z}_m$ which is injective in each partite set and has the property that the values $f(v) - f(u)$ with $u \in A$ and $v \in B$ are different for distinct edges. It is shown in [11] that if H admits a modular bigraceful map then it cyclically decomposes $K_{m,m}$.

To prove theorems 1 and 2 we shall show that a tree T with m edges can be embedded in a tree of the stated size which admits either a modular bigraceful labeling or a ρ-valuation. One of the ingredients of our proofs is the polynomial method of Alon [1]. In particular we shall use the following theorem of Alon:

Theorem 3 (Alon, [1])**.** *Let F be an arbitrary field, and let $f = f(x_1, \cdots, x_n)$ be a polynomial in $F[x_1, \cdots, x_n]$. Suppose the degree $deg(f)$ of f is $\sum_{i=1}^n t_i$, where each t_i is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^n x^{t_i}$ in f is nonzero. Then, if $S_1, \ldots, S_n$ are subsets of F with $|S_i| > t_i$, there are $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ such that*

$$f(s_1, \cdots, s_n) \neq 0.$$

Applications of the polynomial method to other related graph labeling problems can be found in [4,7–9]. We also use the well-known theorem of Kneser. Recall that the stabilizer (or period) of a subset C in an abelian group G is defined by $H(C) = \{g \in G : C + g = C\}$. In other words, $H = H(C)$ is the largest subgroup of G which verifies $H + C = C$. In particular, if G is finite, $|H|$ divides both $|G|$ and $|C|$, since $H(C)$ is a subgroup of G and C is a union of cosets of this subgroup.

Theorem 4 (Kneser, see e.g. [12])**.** *Suppose that A and B are finite non-empty subsets of an abelian group satisfying $|A + B| \leq |A| + |B| - 1$. Then if H is the stabilizer of $A + B$ we have*

$$|A + B| = |A + H| + |B + H| - |H|.$$

Next lemma based on Kneser's Theorem will be used later to prove the existence of appropriate labelings.

Lemma 1. *Let r be a positive integer and let X_1, X_2, Y be non-empty subsets of $\mathbb{Z}_r$ with $|X_1| \geq |X_2|$ and $|Y| > 1$. If the following condition holds*

$$r - |X_1| - |X_2| = |Y| - 1, \tag{1}$$

then $|X_1 + Y| > |X_2|$.

Proof. If $|X_1 + Y| \leq |X_2|$. Then we must have $|X_1 + Y| = |X_2| = |X_1| < |X_1| + |Y| - 1$. By the Kneser's Theorem,

$$|X_1 + Y| = |X_1 + H| + |Y + H| - |H|,$$

where H is the stabilizer of $X_1 + Y$. From this relation and $|X_1 + Y| = |X_1|$ we deduce that $|Y + H| = |H|$ and therefore $|Y| \leq |H|$.

Now, since $|H|$ divides the left hand-side of (1), $|H|$ must also divide $|Y| - 1$. Finally, $|Y| > 1$ implies that $|H| \leq |Y| - 1$, contradicting $|Y| \leq |H|$. □

3. Proof of Theorem 1

Here we consider an extension of the modular bigraceful labeling defined by Cmara, Llado? and Moragas [4], which takes values in an arbitrary abelian group. Let H be a bipartite graph with partite sets A and B, and let $(G, +)$ be an abelian group. A map $f : A \cup B \to G$ is *G-bigraceful* if the restrictions of f to each partite set are injective maps, and the induced values of f over the edges of H are distinct, where again the induced value on an edge $e = uv$ with $u \in A$ and $v \in B$ is $f(v) - f(u)$. Note that a modular bigraceful labeling of a tree with m edges is a $\mathbb{Z}_m$-bigraceful labeling.

We first show that a tree T which admits a $\mathbb{Z}_n$-bigraceful map can be embedded in a tree with n edges which decomposes $K_{n,n}$.

Lemma 2. *Every tree T which admits a $\mathbb{Z}_n$-bigraceful map with n odd is a subtree of a tree T' with n edges that admits a modular bigraceful labeling.*

Proof. Let m be the number of edges of T. Let f be a $\mathbb{Z}_n$-bigraceful map of T. We clearly have $n \geq m$. We define a sequence of trees $T = T_m, T_{m+1}, \ldots, T_n = T'$ by adding one leaf at each step and extend f on T' as a modular bigraceful map.

Suppose we have defined T_i and a $\mathbb{Z}_n$-bigraceful map f on T_i for some $n > i \geq m$. Let A_i and B_i be the two partite sets of T_i with $|A_i| \geq |B_i|$ (we can suppose that by exchanging f for $f_r = n + 1 - f$ if necessary). Let $A'_i = f(A_i)$, $B'_i = f(B_i)$ and $C_i = \{f(y) - f(x) : xy \in E(T_i), x \in A_i, y \in B_i\}$ and $D_i = \mathbb{Z}_n \setminus C_i$. Since T_i is a tree, we have the following relation between these sets:

$$|A_i| + |B_i| = n - |D_i| + 1. \tag{2}$$

It suffices to prove that $|D_i + A'_i| > |B_i|$. In this case there is some $d \in D_i$ and some $a \in A'_i$ such that $d + a \in \mathbb{Z}_n \setminus B'_i$. Define $T_{i+1} = T_i + e_{i+1}$ where e_{i+1} joins the vertex in A_i labeled a to a new vertex v_{i+1} and $f(v_{i+1}) = d + a$, which gives the extension of f to T_{i+1}.

We have that $|D_i| = n - |C_i| = n - i \geq 1$ so either $|D_i| = 1$ which implies as $n = |A_{n-1}| + |B_{n-1}|$ is odd that $|D_{n-1} + A'_{n-1}| = |A_{n-1}| > |B_n - 1|$ or $n - i > 1$. But then we apply Lemma 1 with $r = n$, $X_1 = A'_i$, $X_2 = B'_i$ and $Y = D_i$. The condition (1) of the lemma holds by (2). □

In view of Lemma 2, to prove Theorem 1 (i) it suffices to show that a tree T admits a $\mathbb{Z}_n$-modular bigraceful labeling with some odd $n \geq 2m-1$. Since a star with m edges clearly decomposes $K_{m,m}$, the next Lemma shows that this is indeed the case.

Lemma 3. *Every tree T with m edges admits a $\mathbb{Z}_n$-bigraceful map for each $n \geq m+\max\{|A|,|B|\}-1$, where A and B are the partite sets of T.*

Proof. The proof is by induction on m, the result being obvious for $m=1$. Let $e=uv$ be a leaf of T, where we may assume that $u \in A$ has degree one in T, and let f be a $\mathbb{Z}_n$-bigraceful map on $T'=T-e$ with $n \geq m+\max\{|A|,|B|\}-1$. Let $C=\{f(y)-f(x) : xy \in E(T'), x \in A, y \in B\}$ and $D=\mathbb{Z}_n \setminus C$. Since $|f(v)-D|=|D|=n-m+1 \geq |A|$, there is $d \in D$ such that $f(v)-d \notin f(A\setminus\{u\})$ and we can extend f to T by defining $f(u)=f(v)-d$ resulting in a $\mathbb{Z}_n$-modular bigraceful labeling of T. □

Statement (ii) of Theorem 1 may give a better upper bound for the minimum n for which we can ensure that there is a tree T' with n edges containing a given tree T with the property that T' decomposes $K_{n,n}$. We use the following simple lemma.

Lemma 4. *A tree T with partite sets A and B, $|A| \geq |B|$, has at least $|A|-|B|+1$ leaves in A.*

Proof. Let $A' \subset A$ be the set of non leaves in A and let $T'=T-(A\setminus A')$. Then $|A'|+|B|-1 = |E(T')| = \sum_{x\in A'} d(x) \geq 2|A'|$. Hence $|A'| \leq |B|-1$ and T has at least $|A|-|A'| \geq |A|-|B|+1$ leaves in A. □

Lemma 5. *Let T be a tree with m edges. If p is a primer such that $p \geq \lceil 3m/2 \rceil$, then there is a $\mathbb{Z}_p$-bigraceful map of T.*

Proof. Let A and B be the partite sets of T with $|A| \geq |B|$. By Lemma 4 there is a set $A_0 \subset A$ of leaves such that $|A'|=|A\setminus A_0|=|B|$. Let $T'=T-A_0$. Since $|B| \leq \lceil m/2 \rceil$ and $p \geq m+|B|$ it follows from Lemma 3 that there is a $\mathbb{Z}_p$-bigraceful map f' of T'. Let C' denote the set of edge values of f'. Thus C' is a subset of $\mathbb{Z}_p$ of cardinality $2|A'|-1$.

Let $A_0=\{a_1,\dots,a_k\}$ and let $b_{\sigma(i)}$ be the vertex in B adjacent to a_i, $1 \leq i \leq k$. Consider the polynomial $P \in \mathbb{Z}_p[z_1,\dots,z_k]$ defined as

$$P(z_1,\dots,z_k)=\prod_{1\leq i<j\leq k}(z_i-z_j)\prod_{1\leq i<j\leq k}(b'_{\sigma(i)}-z_i-(b'_{\sigma(j)}-z_j))\prod_{1\leq i\leq k}\prod_{a\in A'}(b'_{\sigma(i)}-z_i-a'),$$

where $b'_{\sigma(i)}=f'(b_{\sigma(i)})$ and $a'=f'(a)$. We can write

$$P=(-1)^{k(k-1)/2+|A'|}\prod_{1\leq i<j\leq k}(z_i-z_j)^2\prod_{1\leq i\leq k} z_i^{|A'|} + \text{ terms of lower degree.}$$

It is known that $\prod_{1\leq i<j\leq k}(z_i-z_j)^2$ has a monomial $z_1^{k-1}\cdots z_k^{k-1}$ with coefficient $\pm k!$; see e.g. [2]. Therefore P has a monomial of maximum degree

$$z_1^{k+|A'|-1}\cdots z^{k+|A'|-1},$$

with nonzero coefficient. Let $D=\mathbb{Z}_p\setminus C'$. Note that $|D|=p-|C'| \geq \lceil 3(|2|A'|+k-1)/2\rceil-2|A'|+1 \geq |A'|+k$. By Alon's Theorem 3, there are $d_1,\dots,d_k \in D$ such that $P(d_1,\dots,d_k)\neq 0$. Extend f' on T' to f on T by defining $f(a_i)=f'(b_{\sigma(i)})-d_i$. Since $\prod_{1\leq i\leq k}\prod_{a\in A'}(b_{\sigma(i)}-d_i-a')\neq 0$, the values of f on A_0 are different from the ones on A'; since $\prod_{1\leq i<j\leq k}(b'_{\sigma(i)}-d_i-(b'_{\sigma(j)}-d_j))\neq 0$, these values are pairwise distinct; finally, since $\prod_{1\leq i<j\leq k}(d_i-d_j)\neq 0$, the edge values $d_1,\dots,d_k$ on the edges incident to $a_1,\dots,a_k$ are pairwise distinct and, since $d_i \in \mathbb{Z}_p\setminus C'$, they are different from the ones taken by f on T'. Thus f is a $\mathbb{Z}_p$-bigraceful map of T. □

Theorem 1 (ii) follows from Lemma 5 and Lemma 2, and using the cyclic decomposition from [11].

4. Extension to ρ -valuation

Following the ideas of the proof of Theorem 1, we give an upper bound for the number of edges that have to be added to an arbitrary tree T to obtain a tree that admits a ρ-valuation in terms of the size of T.

For our present purposes we define a relaxation in the definition of a rho-labeling. Given a graph H on m edges and given $n > m$, a ρ_n-valuation is an injection $\rho_n : V(H) \to \mathbb{Z}_{2n+1}$ such that the induced edge labels defined as before (but now taking the differences modulo $2n+1$) are pairwise distinct.

Lemma 6. *Every tree T of size $m \geq 1$ has a ρ_n-valuation, where $n = \lceil \frac{3m-1}{2} \rceil$.*

Proof. Let $T = T_m, T_{m-1}, \ldots, T_1 = \{v_0v_1\}$ be a sequence of trees such that $T_{i+1} = T_i + v_{i+1}u$ for some $u \in V(T_i)$ and $e_{i+1} = v_{i+1}u$ is a leaf of T_{i+1}. Define a ρ_n-valuation of T inductively as follows. Define $f(v_0) = x_0 \in \mathbb{Z}_{2n+1}$, $f(v_1) = x_1 \in \mathbb{Z}_{2n+1}$ arbitrarily, with $x_0 \neq x_1$. Suppose f is defined on T_i, $1 \leq i < m$, and denote by $V_i = f(V(T_i))$, $C_i = \{\pm(f(x) - f(y)) : xy \in E(T_i)\} \cup \{0\}$ and $D_i = \mathbb{Z}_{2n+1} \setminus C_i$. Since $|D_i + f(u)| = |D_i| = 2n + 1 - 2i - 1 \geq m + 1 > |A_i|$, there is $d \in D_i$ such that $d + f(u) \in \mathbb{Z}_n \setminus V_i$. Thus it is that we can define $f(v_{i+1}) = d + f(u)$ and we eventually get a ρ_n-valuation f on $T_m = T$. □

Lemma 7. *Every tree T of size m that admits a ρ_n-valuation for $n > m$, can be embedded into a tree T' of size n that admits a ρ-valuation.*

Proof. Let f be the ρ_n-valuation of T. We define a sequence of trees $T = T_m, T_{m+1}, \ldots, T_n = T'$ by adding one leaf at each step and extend f on T' as a ρ-valuation.

Suppose we have defined T_i and a ρ_n-valuation f on T_i for some $n > i \geq m$. Denote by $V_i = f(V(T_i))$, $C_i = \{\pm(f(x) - f(y)) : xy \in E(T_i)\} \cup \{0\}$ and $D_i = \mathbb{Z}_{2n+1} \setminus C_i$.

Since T_i is a tree, we have the following relation:

$$2|V_i| - 1 = 2n + 1 - |D_i| \tag{3}$$

Since $|D_i| = 2n + 1 - |C_i| = 2n - 2i \geq 2$ we can apply Lemma 1 with $r = 2n + 1$, $X_1 = X_2 = V_i$ and $Y = D_i$ to obtain that $|D_i + V_i| > |V_i|$. The condition (1) of the lemma holds by (3). Therefore there is some $d \in D_i$ and some $a \in V_i$ such that $d + a \in \mathbb{Z}_n \setminus V_i$. Define $T_{i+1} = T_i + e_{i+1}$ where e_{i+1} joins the vertex in V_i labeled with a to a new vertex v_{i+1}. By defining $f(v_{i+1}) = d + a$ we extend f as a ρ_n-valuation of T_{i+1}. By iterating this procedure we eventually get a ρ-valuation of a tree T' which contains T as a subtree. □

Theorem 2 is a direct consequence of the lemmas 6 and 7, and the fact that a graph with m edges cyclically decomposes K_{2m+1} if and only if it admits a Cvaluation (Rosa [14]).

Another related result is given by Van Bussel [3, Theorem 1]; it implies that every tree with m edges has a ρ-valuation, with $n = 2m - \text{diam}(T)$. Since a random tree has diameter of order $\sqrt{n}$, this lower bound is in general worse than the one obtained in Theorem 2 (see also Lemma 6).

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References

[1] N. Alon, Combinatorial Nullstellensatz, Combin. Probab. Comput. 8 (12) (1999) 7-29.
[2] N. Alon, Additive Transversals, *Israel J. of Mathematics* 117 (2000) 125–130.
[3] F. Van Bussel, Relaxed graceful labellings of trees, *The Electronic Journal of Combinatorics* **9** (2002) # R4.
[4] M. Cámara, A. Lladó, J. Moragas, On a Conjecture of Graham and Häggkvist with the Polynomial Method, *Electronic Notes in Discrete Mathematics* **29C**, 559–563.
[5] J. A. Gallian, A Dynamic Survey of Graph Labeling, *The Electronic Journal of Combinatorics* **5** (2007) # DS6.
[6] R. L. Häggkvist, Decompositions of Complete Bipartite Graphs, *Surveys in Combinatorics*, Johannes Siemons Ed., Cambridge University Press (1989) 115–146.
[7] D. Hefetz, Anti-magic graphs via the Combinatorial Nullstellensatz, *Journal of Graph Theory* **50** (4) (2005) 263?272.
[8] A. E. Kézdy, ρ-Valuations for some stunted trees, *Discrete Mathematics* **306**, 2786–2789.
[9] A.E. Kézdy, H.S. Snevily, Distinct sums modulo n and tree embeddings, *Combinatorics, Probability and Computing* **11** (1) (2002) 35–42.
[10] A. Kotzig, On certain vertex-valuations of finite graphs, Utilitas Math. 4 (1973) 261-290.
[11] A. Lladó, S.C. López, Edge-Decompositions of $K_{n,n}$ Into Isomorphic Copies of a Given Tree, *J. Graph Theory* **48** (2005), no. 1, 1–18.
[12] M. Nathanson, Additive Number Theory: Inverse Theorems and the Geometry of sumsets, GTM Springer 66, New York 1999.
[13] A. Rosa, On certain valuations of the vertices of a graph, Theory Graphs (International Symposium, Rome, 1966) (1967) 349-355.
[14] G. Ringel, Problem 25, in: Theory of Graphs and Its Applications, (Proc. Symp. Smolence, 1963), Czech. Acad. Sci, 1964, p. 162.

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